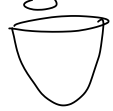
Riemannian mflds & isometry.

Recall, as manifolds,  is equivalent to (diffeomorphic) .

This is obviously un-natural for many applications, eg metal.

One way to get a more rigid notion is to say

$$S^d \subseteq \mathbb{R}^n \simeq \tilde{S}^l \subseteq \mathbb{R}^m$$

only if $n = m$ and there is a Euclidean transformation

$x \mapsto Ax + b$, $A \in O(n)$

taking S to $\tilde{S}$.

We will revisit this — 2^{nd} fundamental form.

But for, eg paper, there is something in between, bending without stretching/isometry. 2 ways to capture this concept turn out to be $\cong$

① A metric space is a set X w/ a "distance" function

$$d : X \times X \to \mathbb{R}$$

s.t.
- $d(x,y) \geq 0$ w/ equality iff $x = y$
- $d(x,y) = d(y,x)$
- triangle inequality: $d(x,y) + d(y,z) \geq d(x,z)$

Rmk: (X,d) has a natural topology.

Eg $\mathbb{R}^n$, $d(x,y) = \sum (x^i - y^i)^2$

If $\gamma : [a,b] \to \mathbb{R}^n$ is smooth, its length is

$$L(\gamma) = \int_a^b \sqrt{\left\|\frac{d\gamma}{dt}\right\|^2}\, dt$$

(Recall Riemann sums)

If $\gamma(t) = (x^i(t))$

$$\sqrt{\sum \frac{dx^i}{dt}^2}$$

Then $d(x,y) = \inf_{\substack{\gamma(a)=x \\ \gamma(b)=y}} L(\gamma)$ (in fact, the min is attained)

(Length space)

If $S^d \subseteq \mathbb{R}^n$ is a manifold, define

$$d(x,y) = \inf_{\substack{\gamma(a)=x \\ \gamma(b)=y \\ \gamma \subseteq S \\ \text{piecewise } C^\infty}} L(\gamma) \leftarrow \Delta \text{ inequality}$$

Note $d^S \geq d^{\mathbb{R}^n} \Rightarrow d$ separating. assume ~~d~~ ~~Reparam~~ parametrization

So, we could say $\varphi: S^d \to \check{S}^d$ is an isometry if it is an isometry of metric spaces. (No longer requires $n = m$)

(2) A priori stronger: It could preserve the length of every curve. but actually easier to check!

Prop $\varphi: S \to \check{S}$ preserves lengths of curves iff $\|d\varphi(v)\|^2 = \|v\|^2$ for all $(p, v_p) \in TS$.

$\Gamma \quad \Leftarrow \quad \frac{d\gamma}{dt}(t) \in T_{\gamma(t)} S$

$\Rightarrow \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} L(\gamma|_{[a,\varepsilon]})$ $\lrcorner$

How to check?

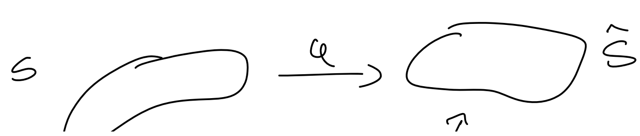
Let ψ be any chart for S near p. $\boxed{\text{define } \langle \, , \, \rangle}$

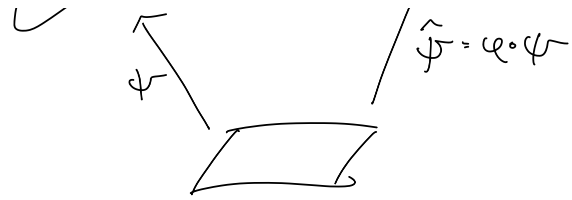
$$\left\| \frac{d}{dt}(\psi \circ x^i) \right\|^2 = \left\langle \frac{\partial\psi}{\partial x^a} \frac{dx^a}{dt}, \frac{\partial\psi}{\partial x^b} \frac{dx^b}{dt} \right\rangle$$

If we let $g_{ab} = \left\langle \frac{\partial\psi}{\partial x^a}, \frac{\partial\psi}{\partial x^b} \right\rangle$ then

$$\left\langle \frac{\partial\psi}{\partial x^a} \frac{dx^a}{dt}, \frac{\partial\psi}{\partial x^b} \frac{dx^b}{dt} \right\rangle = g_{ab} \frac{dx^a}{dt} \frac{dx^b}{dt}$$

Upshot φ preserves lengths of curves / tang. vects iff $\forall$ charts ψ of S,





$g_{ab} = \hat{g}_{ab}$

Gauss — $\{g_{ab}\}$ is the first fund form of the parametrization ψ

Ex $\psi = id : \mathbb{R}^2 \to \mathbb{R}^2$

$$\varphi(x^1, x^2) = \begin{bmatrix} \cos x^1 \\ \sin x^1 \\ x^2 \end{bmatrix}$$

$$g_{11} = \left\langle \frac{\partial \varphi}{\partial x^1}, \frac{\partial \varphi}{\partial x^1} \right\rangle = \left\langle \begin{bmatrix} -\sin x^1 \\ \cos x^1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\sin x^1 \\ \cos x^1 \\ 0 \end{bmatrix} \right\rangle = 1$$

$g_{12} = 0$
$g_{21} = 0$
$g_{22} = 1$

Thm ① $\Longrightarrow$ ②
($\Longleftarrow$ is obvious)

proof to come.

Riemann's insight You can make g_{ab} almost whatever you want!

(but need to be careful if have many charts)

Rmk: Better to restrict to connected manifolds.

Dfn φ is a Riemannian isometry if a diffeo and $\|d\varphi(v)\|^2 = \|v\|^2$ $\forall (p,v) \in TS$.

Linear algebra review

- V vector spaces, $V^* = \mathrm{Hom}(V, \mathbb{R})$

note F injective $\Leftrightarrow$ F^* surjective

ex if $x^1 \ldots x^n$ linear coords on V, then $x^i \in V^*$

- $F: V \to W$ induces $F^*: W^* \to V^*$ (adjoint)
- $B: V \times W \to \mathbb{R}$ bilinear if linear in each variable, these give a vector space $\mathrm{Bil}(V,W)$ say
- $\mathrm{Bil}(V,W)$ is also written $\underline{V^* \otimes W^*}$
  - formal symbols $\ell \otimes m$ $\ell \in V^*$, $m \in W^*$
  - modulo bilinearity. ($\ell(v) m(w)$)
  - basis $v^i \otimes w^j$

- $\mathrm{SymBil}(V) \subseteq \mathrm{Bil}(V,V)$

omit

- $\mathrm{Sym}^2(V^*)$ gen by $\ell\ell'$ modulo bilinearity and swapping defines a quadratic form on V (by $v \mapsto \ell(v)\ell'(v)$) (polynomial)
- It is standard in geometry to identify

$$\mathrm{Sym}^2(V^*) \longrightarrow \mathrm{SymBil}(V)$$
$$\ell\ell' \longmapsto \tfrac{1}{2}(\ell \otimes \ell' + \ell' \otimes \ell)$$

- Positive definite
- $F: V \to W$ pulls back bilinear forms, F^* (need an embedding!)

Cotangent bundle (A quotient ⟶ cvm.

Prop $\exists$ natural mfld str on $\bigcup T_p^* S$ ⇝ cotang. bund.

⌜ ① For open submflds of $\mathbb{R}^n$
② Since local, enough to
Define it in a chart, check consistent ⌟

If $p \in S^d \subseteq \mathbb{R}^n$, have $T_p S \hookrightarrow \mathbb{R}^n$
so $(\mathbb{R}^n)^* \twoheadrightarrow T_p^* S$
ker? all lin fns vanishing on $T_p S$
(conormal bundle $N_p^* S$)

If $f : S \to \mathbb{R}$, then df is a section of T^*S

More about vb

Defn (Isomorphism of v.b.)

Prop • Every vb is locally trivial
• Conversely, if $R \subseteq S \times S$; $\pi_1^* V \xrightarrow{\sim} \pi_2^* V$ ⇝ vb on S/R

Prop If E, E_i's a vb, E' a subbund, then
E/E', E^*, $E_1 \otimes E_2$, etc are naturally vb's.
⌜ Mfld str defined by local triv ⌟